1 Introduction: Initial Definitions and Some Example PDEs

In this course we only deal with problems involving a single partial differential equation (abbreviated pde or PDE in the Notes). A scalar pde is an equation involving an unknown function of two or more independent variables, along with at least two different types of partial derivatives of the unknown function. For example,

$$F(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y), \frac{\partial^2 u}{\partial x^2}(x, y), \ldots) = 0$$

As ordinary differential equations is an extension of (single-variable) integral calculus, partial differential equations can be considered at this level as an extension of multivariable calculus (and ordinary differential equations). Thus, it deals with various operators from calculus (see Appendix B). This includes

$$grad\ \phi(= \nabla\phi) = (\tfrac{\partial\phi}{\partial x}, \tfrac{\partial\phi}{\partial y}, \tfrac{\partial\phi}{\partial z}) \text{ or } (= \tfrac{\partial\phi}{\partial x}\vec{i} + \tfrac{\partial\phi}{\partial y}\vec{j} + \tfrac{\partial\phi}{\partial z}\vec{k})$$

where $\vec{i}, \vec{j}, \vec{k}$ are the unit normal vectors in the x, y, z directions, respectively. (This is expressed here in cartesian coordinates, but in the course we will, briefly, look at equations in polar, cylinder, and spherical coordinate systems.)

If $\vec{F} = (f_1, f_2, f_3)$ is a vector function on $\mathbb{R}^3$ to $\mathbb{R}^3$ (i.e. a vector field), we also have the divergent and curl operators:

$$div\ \vec{F}(= \nabla \cdot \vec{F}) = \frac{\partial}{\partial x}f_1 + \frac{\partial}{\partial y}f_2 + \frac{\partial}{\partial z}f_3$$

$$curl\ \vec{F}(= \nabla \times \vec{F}) = (\frac{\partial}{\partial y}f_3 - \frac{\partial}{\partial z}f_2, \frac{\partial}{\partial z}f_1 - \frac{\partial}{\partial x}f_3, \frac{\partial}{\partial x}f_2 - \frac{\partial}{\partial y}f_1)$$

This allows us to form the important *Laplace* operator

$$div\ (grad) = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

which gives us an equation we will study later in the course, namely **Laplace's equation**:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1}$$

One way of interpreting u satisfying, say, the two-dimensional version of this equation (so $\partial^2 u/\partial z^2 = 0$) is to think of this equation being defined on a bounded domain $\Omega \subset \mathbb{R}^2$, with $u = f(x, y)$ on the boundary of Ω. Then you might think of $u(x, y)$ as the height of an elastic membrane above or below the plane at $(x, y) \in \Omega$.

1.1 Some Other Important Examples

From a historical perspective one of the first equations to be developed and analyzed was the equation for the vibration of a string in a plane, namely

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{2}$$

Here $u(x, t)$ represents the displacement of the string (centerline) from the horizontal at location x, at time t. The constant c is a wave speed parameter. The multi-dimensional version of this equation for surfaces is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0. \tag{3}$$

One of the most important systems of equations ever developed are **Maxwell's equations** linking electric and magnetic fields. Let $\vec{E} = (E_1, E_2, E_3)$ be the electric field vector function (vector field), and $\vec{H} = (H_1, H_2, H_3)$ be the magnetic vector field. There are a number of parameters in this theory, all of which we will treat as constants in my representation here, for convenience. These are μ (magnetic permeability), ρ (charge density), ϵ (dielectric constant), σ (conductivity). There is also the vector field $\vec{J}$, the current density. Then Maxwell's equations can be written in the form

$$\nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \vec{H} \tag{4}$$

$$\nabla \cdot \vec{H} = 0 \tag{5}$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial}{\partial t} \vec{E} + \vec{J} \tag{6}$$

$$\nabla \cdot \vec{E} = \rho/\epsilon \tag{7}$$

Equation (6) is sometimes referred to as Faraday's law, and equation (7) is called Coulomb's law. In electrostatics the quantities are time-independent (We will refer to this in other contexts as the *steady state* situation), so time derivatives are set to zero. Hence, from (4), $\nabla \times \vec{E} = 0$ ($\vec{E}$ is an "irrotational" field). In this case, from calculus, there exists an electrical potential ϕ such that $\vec{E} = -\nabla\phi$. Thus, from (7), $div\ \vec{E} = -div\ (grad\ \phi) = \rho/\epsilon$; that is,

$$\nabla^2\phi = -\rho/\epsilon. \tag{8}$$

This is a *non-homogeneous* version of (1) because of the non-zero right-hand side not involving the dependent variable ϕ; non-homogeneous Laplace's equation is actually referred to as **Poisson's equation**. If the density $\rho = 0$, the Laplace's equation is recovered for ϕ. We will solve and study properties of the solutions to both Laplace's equation and Poisson's equation later in the course. By the way, if $\vec{J} = 0$, then there exists a magnetic potential ψ such that $\vec{H} = -\nabla\psi$. Hence, from (5),

$$\nabla^2\psi = 0.$$

Definition: Any function that satisfies Laplace's equation in some (spatial) domain Ω is called a **harmonic function** in Ω.

In electrodynamics we maintain time dependent changes in the fields. Taking the curl of (4), and utilizing (6), gives

$$\nabla \times \nabla \times \vec{E} = -\mu\frac{\partial}{\partial t}(\epsilon\frac{\partial}{\partial t}\vec{E} + \vec{J}).$$

There is a vector identity given by

$$curl(curl\ \vec{F}) = grad(div\ \vec{F}) - div\ grad\ \vec{F},$$

so

$$\nabla \times \nabla \times \vec{E} = \nabla\ (\nabla \cdot \vec{E}) - \nabla^2\vec{E} = \nabla(\rho/\epsilon) - \nabla^2\vec{E} = -\mu\epsilon\frac{\partial^2}{\partial t^2}\vec{E} - \mu\frac{\partial}{\partial t}\vec{J}$$

or

$$\mu\epsilon\frac{\partial^2}{\partial t^2}\vec{E} + \mu\sigma\frac{\partial}{\partial t}\vec{E} - \nabla^2\vec{E} = -\frac{1}{\epsilon}\nabla\rho.$$

Let $c^2 := 1/\epsilon\mu$, $\gamma := \sigma/\epsilon$, $\vec{F} := -\frac{1}{\mu\epsilon^2}\ \nabla\rho$, then

$$\frac{\partial^2}{\partial t^2}\vec{E} + \gamma\frac{\partial}{\partial t}\vec{E} = c^2\nabla^2\vec{E} + \vec{F}.$$

That is, each component of $\vec{E}$ (and it turns out this holds for each component of $\vec{H}$ also) satisfies an equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = c^2\nabla^2 u + f.$$

If $\rho =$ constant, then this equation reduces to

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = c^2\nabla^2 u. \tag{9}$$

A one-dimensional version of it is a version of the **telegrapher's equation**. We will discuss this equation when we discuss *dissipation* and *dispersion* ideas. Note that if the conductivity $\sigma = 0$, then (9) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

This is the standard **wave equation** given in (3). If you consider for a moment that the left-hand side of equation (9) is associated with a 'damped spring', and the right-hand side of the equation deals with the spring force, then you should know from your ODE class that the γ term acts to dampen the spring motion. Put another way, in comparing (3) and (9), the γ term in (9) acts to dissipate energy in the moving string or surface described by u. Hence, we refer to (9) as a *damped* wave equation.

Remark: An active research topic these days is how to design a cloak, that is, designing a way of not only concealing an object, or person, from view (visible light, electromagnetic rays, etc.), but doing it in such a way that observers do not know an object is being hidden. Think of Harry Potter's way of creating an invisible cloak to escape notice of Draco Malfoy and other Death Eaters, or the ability of Romulan ships to cloak in the "Star Trek" TV series (and movies). There is a nice article on this topic using Laplace's equation instead of Maxwell's equations by K. Bryan and T. Leise (SIAM Review, vol. 52, 2010, pp 359-377).

Another class of problems we will discuss extensively involve, as a prototypical case, the **heat equation**

$$\frac{\partial u}{\partial t} = D\nabla^2 u. \tag{10}$$

Here D is a **diffusivity** constant (property of the material being heated), and $u(\vec{x}, t)$ is the thermal energy at location $\vec{x}$, at time t. In the case of one dimension, with specified distributed internal 'heat sources' f, (10) becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x, t).$$

This is a model for a long, thin rod where with lateral sides being completely insulated, and each cross-section of the rod being isothermal, so variation in temperature only depends on location x along the rod. There is also the assumption that the rod has uniform material properties so that D is a constant.

Remark: I will discuss cases where the temperature of the rod reaches a **steady state** temperature (on some spatial domain); this implies we are considering the case of the left-hand side of the equation being set to zero, and the source function f being independent of t, so

$$D\frac{\partial^2 u}{\partial x^2} + f(x) = 0.$$

This is another case of using ODE methods to discuss a class of solutions for the heat equation. Note that this is a one-dimensional Poisson equation.

Here are a few more examples to consider:

Pricing of financial derivative contracts: Much discussion in the press about the need for government oversight of financial institutions concerns the regulation of option contracts (financial derivatives). The simplest of these is a call or put option on an underlying asset (e.g. a stock, a currency, a piece of real estate, or even another financial derivative). Such an option is a contract giving the holder the right, but not the obligation, to buy (or sell) the underlying asset, at some specified future time, when the value of the underlying asset reaches, or exceeds, a value called the exercise price. For a given time t during the life of the contract, and for a given value, S, of the asset, let $V = V(S,t)$ be the value of the contract. In *Black-Scholes* theory, V is the solution to the pde

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0 \tag{11}$$

where r is the "risk-free" interest rate, and σ is the "volatility" of the asset (standard deviation of the stochastic underlying asset price). This is the starting point for a large number of valuation problems in financial engineering. It turns out (11) can be reduced down to solving the heat equation through a certain transformation. (This observation is extremely important because it means that security traders could use, and misuse, a formula for the solution $V(S,t)$ without knowing anything about pdes; hence, the notional value of traded options is in the tens of trillions of dollars now.)

Evolution of a quantum state: In quantum mechanics the wave function $\psi(x,t)$ that characterizes the one-dimensional motion of a particle under the influence of the potential $V(x)$ satisfies the *Schrödinger* equation

$$ih\frac{\partial \psi}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \tag{12}$$

where h is (reduced) Planck constant, m is the mass of the particle, and $i^2 = -1$. We will deal with equations in this course that only have real coefficients.

Brittle fracture: In modeling cracks in perfectly elastic solids, the simplest case is to consider the displacement field $\vec{u} = (0,0,w(x,y))$, where w is harmonic. For uniform shearing at large distances ($|y| \to \infty$), $w \to \tau y$, $0 < \tau = constant$; if we let $w = \tau y + u(x,y)$, then a model here is for $\nabla^2 u = 0$, for $y > 0$, $u = 0$ on $y = 0, |x| > c$, and $\partial u/\partial y = -\tau$ on $y = 0, |x| < c$, with $u \to 0$ as $x^2 + y^2 \to \infty$. One must also specify certain conditions at the crack tips $(\pm c, 0)$.

Shallow water theory: In the one-dimensional spatial case of water flow, the velocity function u satisfies the *Korteweg-deVries* equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0 \tag{13}$$

Euler-Bernoulli beam: For a thin beam of modest displacement, negligible rotary inertial, and insignificant stress across any beam section, a common model is the Euler-Bernoulli beam equation

$$\frac{\partial^2}{\partial x^2}\{EI\frac{\partial^2 u}{\partial x^2}\} + \rho\frac{\partial^2 u}{\partial t^2} = f(x,t) \tag{14}$$

where EI is the flexural rigidity (Young's modulus $\times$ moment of inertia), ρ is linear density of the beam, and $f(x,t)$ is a distributed body force. We will discuss this equation more as an example of applying our solution techniques to a higher order equation.

Minimal surfaces in $\mathcal{R}^3$: You studied curvature in your multi-variable calculus class. Mean curvature at a point on a surface is the average normal curvature. A surface $S \subset \mathcal{R}^3$ is *minimal* if and only if its mean curvature vanishes identically.
(Meusnier, 1776): the condition on the mean curvature can be expressed by the PDE

$$(1+u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1+u_y^2)u_x = 0.$$

So a surface is minimal if and only if it can be locally represented as the graph of the solution to this second-order quasilinear elliptic PDE.

Remark: Notice that we used subscripts in the last PDE to imply partial derivatives. This will be a common practice in these Notes.

Digital halftoning: Halftoning involves rendering a normal continuous tone image into an array of black-and-white dots. One of the techniques used is to consider a *backward* diffusion equation

$$\frac{\partial v}{\partial t} = -\nabla\cdot(M(v)\ \nabla v)$$

defined on the image domain, where the *mobility function* $M(v) \geq 0$ is of a special kind (but makes the equation nonlinear). Here $t > 0$ serves as an artificial evolution parameter that is used to control the strength of the operation. Other uses of PDEs in image studies include *inpainting* (e.g. removing scratches from (digitized) chemical photo images, and identifying edges of objects in blurred images, that is *deblurring*.

Flow of a thin layer of paint down a vertical wall: Let x axis increase in the downward direction, $h = h(x,t)$ be the paint thickness at distance x

from the ceiling, and $u = u(x, y, t)$ be the velocity of the fluid (paint). In the case where u is essentially unidirectional (downward), shear force is zero at the paint surface ($\partial u/\partial y = 0$ at $y = h$), and the paint sticks to the wall ($u_{|y=0} = 0$), then from a conservation of mass argument the paint thickness h satisfies

$$\frac{\partial h}{\partial t} + ch^2 \frac{\partial h}{\partial x} = 0 \tag{15}$$

where c is a constant that is proportional to the gravitational constant.

1.2 Additional Definitions and Terminology

Definition: the **order** of an equation is the order of the highest derivative appearing in the equation (just as in the ODE case). For example, (1)-(3), (8)-(12) are each second order, (13) is third order, (14) is fourth order, and (4)-(7), (15) are first order.

Remark: Domains
PDEs are generally associated with some physical, biological, financial, geometric, etc. model situation, so associated with the equation is a domain where the equation is to be specified. For example, for (3) and (14), strings and beams are usually considered of finite length, so in these cases the appropriate domain would be $\Omega = \{(x, t) : 0 < x < L, t > 0\}$. In these cases, in order to have a well-posed problem, it is necessary to impose **boundary conditions** at $x = 0$ and $x = L$, and **initial conditions** at $t = 0$.

Similarly, for Laplace's equation (1), say in two dimensions ($\partial^2/\partial z^2 = 0$), typically we would have a bounded, simply-connected ("no holes") domain, with a piece-wise smooth boundary. That is, there is a normal derivative defined at every point on the boundary, except maybe at a finite number of isolated points (e.g., a polygonal domain).

Sometimes circumstances call for considering a 'spatially' unbounded domain. For example, for (11), the "sky is the limit" on the price of an asset, but it can not go below 0, so $\Omega = \{(S, t) : 0 < S, 0 < t < T = \text{expiration time of the contract}\}$. For the paint problem, (15), $\Omega = \{(x, t) : 0 < x, 0 < t\}$ might be sufficient. In such situations there might be an implicit condition at infinity; that is, contract value $V(S, t)$ might be constrained by growth or boundedness condition as $S \to \infty$, for example. Domains and boundary conditions will be extensively discussed throughout the course. But keep in mind that domains are open sets.

Remark on Classical Solutions
Most of the time in this course we will derive **classical solutions**, that is, functions that are defined on the whole domain, that satisfy the initial and/or boundary conditions, that are *smooth* in the sense that the function has all the derivatives appearing in the equation (and that are continuous on the domain and its boundary), and that the function satisfies the equation in the domain.

There is, however, problems we will study that do not always have classical solutions (because of the physical nature of the problems). We will hint at what is done when we get to such problems, but we will not have time to dwell on the technicalities in this course.

Some Function Spaces
Differential equations, both partial and ordinary equations, hold on **open sets** representing their domains. Derivatives of the solution appearing in the equation are not expected to hold at boundary points. But the solution usually is expected to be continuous in its domain right up to and including the boundary. Thus, if we are considering Laplace's equation, for example,

$$u_{xx} + u_{yy} = 0 \text{ for } 0 < x < 1, 0 < y < 2 \ ,$$

then we expect to have the solution be continuous for $0 \leq x \leq 1, 0 \leq y \leq 2$, and have two continuous derivatives in both variables for $0 < x < 1, 0 < y < 2$. Also, Laplace's equation can be viewed as an operator equation, so we might write $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = 0$ in Ω, or $Lu = 0$ in Ω, where $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Viewed this way L is a function defined on a function space. Then the domain of L needs to be specified. Since $\Omega \subset \mathbb{R}^2$ is an open set (it doesn't contain its boundary points), then its **closure**, $\overline{\Omega}$, does; that is, $\overline{\Omega} = \Omega \cup \partial\Omega$. For Ω, we write $C^n(\Omega)$ to mean the set of functions that are defined on Ω and that have n continuous derivatives (in all variables) in Ω. Then $C^0(\Omega)$ would mean be the set of functions that are continuous in Ω, but in this case we will write $C(\Omega)$ instead of $C^0(\Omega)$. Also, $C^1(\Omega)$ is the set of functions that are continuously differentiable in Ω. Similarly, $C^\infty(\Omega)$ will mean the set of functions with continuous derivatives of all orders in Ω. Hence, a possible domain for the operator L given above would be $C(\overline{\Omega}) \cup C^2(\Omega)$. This is a more modern way of viewing pdes, and it has greatly helped the theory of pdes, but we will not get too carried away with abstraction in this course.

1.3 Some Sample Problems with Solutions

Example: Notice that for any integer n, $u(x, y) = sin(nx)\ sinh(ny)$ is a solution to Laplace's equation $u_{xx} + u_{yy} = 0$.

Example: Find a function $u = u(x, t)$ that satisfies

$$\frac{\partial^2 u}{\partial x^2} = 0 \text{ for } 0 < x < 1, t > 0$$
$$u(0, t) = t^2 \text{ and } u(1, t) = 1 \text{ for } t > 0$$

Solution: $\frac{\partial u}{\partial x} = A = A(t) \rightarrow u(x, t) = A(t)x + B(t)$. Now $u(0, t) = t^2 = B(t)$, and $u(1, t) = 1 = A(t) + t^2$. Therefore, $u(x, t) = (1 - t^2)x + t^2$.

Remark: In ODEs we obtain constants of integration (that are pure constants). **Not** in PDEs. When integrating we obtain arbitrary functions of all the other variables not being the variable of integration. A further example to emphasize this is $u_{xy} = 0$ defined in the x, y plane. Integrating with regard to y gives $u_x = A(x)$; since $A(\cdot)$ is an arbitrary function of x at this point, it is convenient to write $u_x = A'(x)$, where $()'$ means d/dx. Then integrating by x gives $u(x, y) = A(x) + B(y)$, where A and B are considered arbitrary, differentiable functions of a single variable.

1.4 Summary

Here are some short-answer quiz questions:

1. What is the definition of a partial differential equation, and what is meant by its order?
2. What is Laplace's equation? Poisson's equation?
3. What is the difference between the wave equation and the telegrapher's equation?
4. What is the *Schrödinger* equation? The heat equation?
5. What is meant by $u(x, y)$ being a *harmonic* function in set $U \subset \mathbb{R}^2$?
6. What is $C^n(U)$? Give an example of a function that is in $C(0, 1)$, but not in $C^1(0, 1)$.

7. What is meant by a function being a *classical solution* to a PDE?

Exercises

1. Verify that $u(x,y) = ln(\sqrt{x^2+y^2})$ satisfies Laplace's equation.

2. Verify that $\frac{e^{-x^2/4Dt}}{2\sqrt{\pi Dt}}$ satisfies the heat equation $u_t = Du_{xx}$.

3. Verify that $u(x,t) = \int_0^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds$ is a solution to $u_t = u_{xx}$.

4. Verify that $u(x,y) = f(x)g(y)$ is a solution to $u\, u_{xy} = u_x u_y$ for all pairs of differentiable functions f, g of a single variable in $\mathcal{R}$.

5. Verify that the solution $u = u(x,t)$ of the transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\phi(u) = 0, u(x,0) = f(x),$$

   for sufficiently smooth ϕ is implicitly given by $u = f(x - \phi(u)t)$.

6. Return to the Black-Scholes equation, (11), on page 6 and make the following transformation to reduce (11) to the heat equation. First let E be the "exercise price" and write $S = Ee^x$. Also write $t = \tau/(\sigma^2/2)$ and $V = Eu(x,\tau)$. Substitute these into equation (11) and show the equation u solves is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku, \text{ where } k := \frac{r}{\sigma^2/2}$$

   (Here $\frac{\partial}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial}{\partial \tau}, S\frac{\partial}{\partial S} = \frac{\partial}{\partial x}$, and $\frac{\partial^2}{\partial S^2} = -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{1}{S^2}\frac{\partial^2}{\partial x^2}$.)
   Now let $u(x,\tau) = e^{\alpha x - \beta \tau} w(x,\tau)$ and substitute this into the u equation to obtain the equation for w of the form

$$w_\tau = w_{xx} + Aw_x + Bw$$

   where $A = A(\alpha)$ and $B = B(\alpha,\beta)$. Therefore, choose the free parameters α and β such that $A = 0$ and $B = 0$. Later we will solve equations like $w_\tau = w_{xx}$, and by unwrapping these transformations a formula for $V(S,t)$ is obtained.